

THE COMPLEMENT OF THE CLOSED UNIT BALL IN \mathbb{C}^3 IS NOT SUBELLIPTIC

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ABSTRACT. In this short note we show that $\mathbb{C}^n \setminus \overline{\mathbb{B}^n}$ is not subelliptic for $n \geq 3$. This is done by proving a Hartogs type extension theorem for holomorphic vector bundles.

1. INTRODUCTION

Gromov [10] introduced in 1989 in his seminal paper the notion of an elliptic manifold and proved an Oka principle for holomorphic sections of elliptic bundles, generalizing previous work of Grauert [9] for complex Lie groups. The (basic) Oka principle of a complex manifold X is that every continuous map $Y \rightarrow X$ from a Stein space Y is homotopic to a holomorphic map.

In Oka theory exist many interesting classes of complex manifolds with weaker properties than ellipticity. However, all the known inclusion relations between these classes are yet not known to be proper inclusions. Following the book of Forstnerič [2, Chap. 5] we want to mention in particular the following classes and then prove that at least one of these inclusions has to be proper.

Definition 1.1. A *spray* on a complex manifold X is a triple (E, π, s) consisting of a holomorphic vector bundle $\pi : E \rightarrow X$ and a holomorphic map $s : E \rightarrow X$ such that for each point $x \in X$ we have $s(0_x) = x$.

The spray (E, π, s) is said to be *dominating* if for every point $x \in X$ we have

$$d_{0_x}s(E_x) = T_xX$$

A complex manifold is called *elliptic* if it admits a dominating spray.

A weaker notion, subellipticity, was introduced later by Forstnerič [1] where he proved the Oka principle for subelliptic manifolds:

Definition 1.2. A finite family of sprays $(E_j, \pi_j, s_j), j = 1, \dots, m$, on X is called *dominating* if for every point $x \in X$ we have

$$(1) \quad d_{0_x}s_1(E_{1,x}) + \dots + d_{0_x}s_m(E_{m,x}) = T_xX$$

A complex manifold X is called *subelliptic* if it admits a finite dominating family of sprays.

It is immediately clear from the definition that an elliptic manifold is subelliptic. However, the Oka principle holds under even weaker conditions. Forstnerič [3, 4, 5] showed that the following condition (CAP) is equivalent for a manifold to satisfy the Oka principle (and versions of the Oka principle with interpolation and approximation), hence justifying the name Oka manifold:

Definition 1.3. A complex manifold is said to satisfy the *convex approximation property* (CAP) if for on any compact convex set $K \subset \mathbb{C}^n$, $n \in \mathbb{N}$, every holomorphic map $f : K \rightarrow X$ can be approximated, uniformly on K , by entire holomorphic maps $\mathbb{C}^n \rightarrow X$. If this approximation property holds for a fixed $n \in \mathbb{N}$, then X is said to satisfy CAP_n . A manifold satisfying CAP is called an *Oka manifold*.

A subelliptic manifold is always Oka. Whether an Oka manifold is (sub-)elliptic, is on the other hand not known – this implication holds however under the extra assumption that it is Stein.

We want to mention also these weaker properties:

Definition 1.4. A complex manifold X of dimension n is called *dominable* if there exists a point $x_0 \in X$ and a holomorphic map $f : \mathbb{C}^n \rightarrow X$ with $f(0) = x_0$ and $\text{rank } d_0 f = n$.

Definition 1.5. A complex manifold X of dimension n is called *strongly dominable* if for every point $x_0 \in X$ there exists a holomorphic map $f : \mathbb{C}^n \rightarrow X$ with $f(0) = x_0$ and $\text{rank } d_0 f = n$.

Summarizing the previous, the following inclusions are known:

$$\text{elliptic} \subseteq \text{subelliptic} \subseteq \text{Oka} \subseteq \text{strongly dominable} \subseteq \text{dominable}$$

There are several known candidates to prove that one of this inclusions is proper. We will present here an example of complex manifold which is not subelliptic, but strongly dominable. Whether or not it is Oka remains an open question.

Theorem 1.6. *For $n \geq 3$ the manifold $X := \mathbb{C}^n \setminus \overline{\mathbb{B}^n}$ is not subelliptic.*

The proof of this relies on a Hartogs type extension result for holomorphic vector bundles in dimensions $n \geq 3$. This extension result is not valid in dimension 2, so the question as to whether $\mathbb{C}^2 \setminus \overline{\mathbb{B}^2}$ is elliptic or sub-elliptic remains open.

Recently, Forstnerič and Ritter [7] have proved that $X = \mathbb{C}^n \setminus \overline{\mathbb{B}^n}$ satisfies CAP_m for all $m < n$, but it remains an open question whether X is an Oka-manifold.

2. PROOF OF THEOREM 1.6

Assume that $E \rightarrow X$ is a holomorphic vector bundle with a spray $s : E \rightarrow X$. By Corollary 2.4 there exists a point $p \in b\mathbb{B}^n$ and a (small) open ball B centered at p such that $E|_B$ is the trivial bundle $B \times \mathbb{C}^r$. We use (z, w) as coordinates on $B \times \mathbb{C}^r$. On $(B \setminus \overline{\mathbb{B}^n}) \times \mathbb{C}^r$ we may write the spray $s = (s_1, \dots, s_n)$ as

$$s_j(z, w) = \sum_{\alpha} g_{\alpha}^j(z) w^{\alpha},$$

where α is a multi-index. By the Hartogs extension theorem for the functions g_{α}^j we see that the spray extends to some neighborhood of p , and so by possibly having to choose a smaller B we assume that s extends to $B \times \mathbb{C}^r$.

Given a finite number of sprays (E_j, π_j, s_j) we may repeat the argument for all of them simultaneously to find a ball B around a point $p \in b\mathbb{B}^n$ to which the bundles and sprays extend. We claim that the set of points $Z \subset B$ where the family of sprays is not dominating is an analytic set. To see this, choose holomorphic sections $\mathcal{B}_j = \{f_1^j, \dots, f_{k_j}^j\}$ for $j =$

$1, \dots, m$, where $k_j = \dim(E_j)$ such that $\mathcal{B}_{j,z}$ forms a basis for $E_{j,z}$ for all $z \in B$. Let $g_s^j(z) = d_{0_z} s_j(f_s^j(z))$, and let $Z \subset B$ be the set of points in B where the dimension of the span of all the vectors g_s^j is less than n . The set Z has to be a proper analytic subset of B and so there exists a point $q \in b\mathbb{B}^n$ such that one of the sprays, say s_1 satisfies $d_{0_q} s_1(E_{1,z}) \cap \mathbb{B}^n \neq \emptyset$, which implies that $s_{1,z}(E_{1,z}) \cap \mathbb{B}^n \neq \emptyset$. But then $s_1(E_{1,w}) \cap \mathbb{B}^n \neq \emptyset$ for points w close to q which contradicts the assumption that s_1 is a spray into X . \square

The argument given relies on the following extension result for locally free sheaves by Siu [6].

Definition 2.1. For $0 < a < b \in \mathbb{R}^N$ we let $G^N(a, b)$ denote the set

$$G^N(a, b) = \{z \in \mathbb{C}^N : |z_j| < b_j \text{ for } j = 1, \dots, N \text{ and } |z_j| > a_j \text{ for some } 1 \leq j \leq N\}.$$

Proposition 2.2. (Siu, [6], page 144) Suppose $0 \leq a \leq a' < b \in \mathbb{R}^N$ and D is an open subset of \mathbb{C}^n . Suppose S_i is a coherent analytic sheaf on $D \times G^N(a, b)$ such that $S_i = S_i^{[n]}$ ($i = 1, 2$). If $\varphi : S_1 \rightarrow S_2$ is a sheaf-isomorphism on $D \times G^N(a', b)$ then φ can be uniquely extended to a sheaf isomorphism $S_1 \rightarrow S_2$ on $D \times G^N(a, b)$.

Theorem 2.3. (Siu, [6], page 225) Suppose $0 \leq a < b \in \mathbb{R}^2$, D is a domain in \mathbb{C}^n , and S is a locally free sheaf of rank r on $D \times G^2(a, b)$. Suppose A is a thick set in D and, for every $t \in A$ $S(t)$ can be extended to a coherent analytic sheaf on $\{t\} \times \Delta^2(b)$. Then S can be extended uniquely to a coherent analytic sheaf \tilde{S} on $D \times \Delta^2(b)$ satisfying $\tilde{S}^{[n]} = \tilde{S}$.

Corollary 2.4. Let $\pi : E \rightarrow \mathbb{C}^n \setminus \overline{\mathbb{B}}^n$ be a holomorphic vector bundle of rank r . Then there exists a finite set of points $P \subset b\mathbb{B}^n$ such that for any point $p \in b\mathbb{B}^n \setminus P$, there exists an open neighborhood U_p of p such that E extends to a holomorphic vector bundle on U_p .

Proof. Let $p = (1, 0, \dots, 0)$. Let B_ϵ be the ball of radius ϵ centered at p in the plane $\{z_{n-1} = z_n = 0\}$, let $a = (\sqrt{1 - (1 - \epsilon)^2}, \sqrt{1 - (1 - \epsilon)^2})$, and let $b = (1, 1)$. By Theorem 2.3 we get an extension of E from $B_\epsilon \times G^2(a, b)$ to $B_\epsilon \times \Delta^2(b)$ as a coherent analytic sheaf \tilde{E} . By [8] \tilde{E} is a vector bundle outside an analytic set Z , i.e., a finite set of points, since by Proposition 2.2 Z cannot project to the set $\{|z_1|^2 + \dots + |z_{n-3}|^2 > 1\}$. So perhaps after having to choose a different point p and a smaller ϵ , and referring to Proposition 2.2, we may assume that \tilde{E} is a vector bundle on $B_\epsilon \times \Delta^2(b)$. Hence \tilde{E} is trivial, and we may find holomorphic sections $s = (s_1, \dots, s_r)$ of the original bundle $E \rightarrow B_\epsilon \times G^2(a, b)$ generating E . Since s represents an isomorphism between E and the trivial bundle, we may use Proposition 2.2 on different Hartogs figures $B_{\delta(z)}(z) \times G^2(a(z), b)$, $z \in B_\epsilon$ to extend s holomorphically to $E \rightarrow (B_\epsilon \times \Delta^2(b)) \setminus \overline{\mathbb{B}}^3$ (note that $\{z_{n-1} = z_n = 0\}$ is removable for a vector bundle isomorphism). The extension is generating outside a hypersurface, hence everywhere, and so defines an isomorphism with the trivial bundle. \square

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